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Periodic Exponential Box Splines on a Three Direction Mesh

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Multivariate "truncated Tchebycheff" functions which generalize the univariate Green's function on the one hand and the multivariate "truncated powers" on the other hand are constructed and analysed for the three direction mesh. For weight functions which are products of an exponential and a periodic function, the application of suitable divided differences to the corresponding truncated Tchebycheff functions yields a new type of box spline termed "periodic exponential." This class of box splines contains the exponential (hence polynomial) box splines of the three direction mesh as special cases. It also extends the notion of the univariate periodic exponential Tchebycheffian B-splines.

1. INTRODUCTION

One of the main features of univariate Tchebycheffian B-splines is that they can be expressed as linear combinations of the one side supported Green's functions (cf. [S; Th. 9.22]). For polynomial B-splines these Green's functions are the simple truncated powers $(x-\alpha)_+^{n-1}$. Constructing a multivariate version of truncated powers, Dahmen showed in [D] that such a relation holds for simplicial B-splines (see als [DM_1; pp. 23–24]), and it is well known that polynomial box splines can also be obtained from linear combinations of these truncated powers.

In this paper we construct multivariate "truncated Tchebycheff" functions, which generalize the univariate Green's functions on the one hand and the multivariate truncated powers on the other hand. Yet, it is not clear how to obtain a compactly supported element from linear combinations of these truncated Tchebycheff functions. Observing this difficulty we confine our construction to a special class of "truncated" functions. The univariate analog of this class, termed *periodic exponential*, was introduced

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and investigated in [DR_1]. The corresponding B-splines were shown to satisfy a recurrence formula of the form

$$B_n(x) = w_n(x) \int_{x-1}^{x} B_{n-1}(t)\, dt, \tag{1.1}$$

where the weight function $w_n(x)$ is "periodic exponential," namely,

$$w_n(x) = e^{\rho_n x} r_n(x), \qquad \rho_n \in \mathbb{R},\ r_n(x+1) = r_n(x)\ \forall x \in \mathbb{R}. \tag{1.2}$$

Evidently the cardinal polynomial B-splines, as well as the cardinal exponential B-splines, are contained in this class.

Here, we make a first step in the construction of the multivariate box version of the above periodic exponential B-splines. For the three direction mesh in $\mathbb{R}^2$ related to the directions $(1, 0)$, $(0, 1)$, $(1, 1)$, a corresponding periodic exponential box spline (PEB-spline), $B(\mathbf{x})$, defined by a formula similar to (1.1), is introduced, and some of its basic properties are presented.

In Section 2, given a fixed knot $\boldsymbol{\alpha} \in \mathbb{R}^2$, a three direction truncated Tchebycheff, $T(\cdot\,; \boldsymbol{\alpha})$, is defined and is shown to maintain exactly the same smoothness as the corresponding truncated power (see, e.g., [DM_1, p. 17]). In Section 3 the PEB-spline, $B(\cdot\,; \boldsymbol{\alpha})$ is introduced and is shown to be nonnegative with the same support as the polynomial box spline defined by the same directions. The smoothness question of the PEB-spline is settled by showing that $B(\cdot\,; \boldsymbol{\alpha})$ is a linear combination of "truncated periodic exponential" functions. In Section 4 it is proved that, under a certain restriction on the weight functions, the PEB-spline can be expressed as a linear combination with constant coefficients of four lower order PEB-splines.

Other properties of the periodic exponential box spline, as well as the generalization of the results here to multidimensional multidirectional PEB-splines, are still under investigation.

2. The Truncated Tchebycheff

For a three direction mesh in $\mathbb{R}^2$ we construct an analog of the univariate Green's function (see, e.g., [S, Sect. 9.2]). The main result here shows that the smoothness of this function, termed *truncated Tchebycheff*, always coincides with the smoothness of its simplest case, namely the "truncated power" [D].

Let $E = \{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ be the set of $\{(1, 0), (0, 1), (1, 1)\}$, respectively. Let $\Gamma_n = \{\gamma_j\}_{j=1}^{n}$ $(n \geqslant 2)$ be a *defining set* with elements of the form

$\gamma_j = \{\mathbf{x}^i, w_j(t)\}$ where $\mathbf{x}^i \in E$ and $w_j(t)$ is a positive univariate weight function which is infinitely differentiable everywhere. For simplicity of presentation we also assume that

(a) $\mathbf{x}^j = \mathbf{e}^j, j = 1, 2$.

(b) There exists $2 \leqslant m \leqslant n$ such that $\mathbf{x}^j = \mathbf{e}^3$ if and only if $j > m$.

Nevertheless the results obtained in Sections 2, 3 are valid also for a general three directional defining set Γ_n. In what follows, m and n retain their meaning as above.

DEFINITION 2.1. The truncated Tchebycheff related to Γ_n with a knot $\boldsymbol{\alpha} \in \mathbb{R}^2$ is defined inductively as follows,

$$T(\Gamma_2 \mid \mathbf{x}; \boldsymbol{\alpha}) = \begin{cases} w_1(x_1)\, w_2(x_2), & \mathbf{x} - \boldsymbol{\alpha} \in \mathbb{R}^2_+, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

and for $k \geqslant 3$

$$T(\Gamma_k \mid \mathbf{x}; \boldsymbol{\alpha}) = \begin{cases} w_k(x_1) \displaystyle\int_{-\infty}^{x_1} T(\Gamma_{k-1} \mid (t, x_2); \boldsymbol{\alpha})\, dt, & \mathbf{x}^k = \mathbf{e}^1, \\ w_k(x_2) \displaystyle\int_{-\infty}^{x_2} T(\Gamma_{k-1} \mid (x_1, t); \boldsymbol{\alpha})\, dt, & \mathbf{x}^k = \mathbf{e}^2, \\ w_k\left(\dfrac{x_1 + x_2}{2}\right) \displaystyle\int_{-\infty}^{(x_1+x_2)/2} & \\ \quad \times T\left(\Gamma_{k-1} \,\middle|\, \left(\dfrac{x_1 - x_2}{2} + t, \dfrac{x_2 - x_1}{2} + t\right); \boldsymbol{\alpha}\right) dt, & \mathbf{x}^k = \mathbf{e}^3. \end{cases} \tag{2.2}$$

As usual $\mathbb{R}^2_+ = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 \geqslant 0, x_2 \geqslant 0\}$.

It is clear that a similar definition in the univariate case leads to the classical Green's function, [S; Sect. 9.2]. Note that in each case the integration in (2.2) is taken along a ray with a vertex at $\mathbf{x}$ and directed as $\mathbf{x}^k$, hence

PROPOSITION 2.1. *$T(\Gamma_k \mid \mathbf{x}; \boldsymbol{\alpha})$ is a nonnegative function which is positive in the interior of the quadrant $\{\mathbf{x} \mid \mathbf{x} - \boldsymbol{\alpha} \in \mathbb{R}^2_+\}$.*

EXAMPLE 2.1. In the case $w_j(t) \equiv 1, j = 1, ..., n$, the truncated Tchebycheff is the three direction version of the well-known truncated power introduced in [D].

EXAMPLE 2.2. When $w_n(t) = e^{-t}$, $w_{n_j}(t) = e^{-t/2}$, where $n_j = \max\{i \mid \mathbf{x}^i = \mathbf{e}^j\}$, $j = 1, 2$, and all other weight functions are constants, direct computation shows that $T(\Gamma_n | \mathbf{x}; \mathbf{0})$ is the "piecewise exponential" of [DM$_2$] for the three direction case.

Since $T(\Gamma_k | \mathbf{x}; \boldsymbol{\alpha})$ $(3 \leqslant k \leqslant n)$ was obtained by iterated integrations, it can be reduced by a suitable differential operator:

PROPOSITION 2.2. *For every $m < k \leqslant n$ denote*

$$D^k := D_{\mathbf{e}^3} - \frac{w_k'((x_1+x_2)/2)}{w_k((x_1+x_2)/2)}, \tag{2.3}$$

where $D_{\mathbf{x}}$ denotes, as usual, the $\mathbf{x}$ directional derivative. Then

$$D^k T(\Gamma_k | \cdot; \boldsymbol{\alpha})|_{\mathbf{x}} = w_k\left(\frac{x_1+x_2}{2}\right) T(\Gamma_{k-1} | \mathbf{x}; \boldsymbol{\alpha}). \tag{2.4}$$

Using (2.2) *the verification of* (2.4) *is straightforward. Of course, analogous relations to* (2.4) *hold for* $3 \leqslant k \leqslant m$.

As mentioned before, our main aim here is to study the smoothness of $T(\Gamma_k | \cdot; \boldsymbol{\alpha})$. This question is easily settled for $k \leqslant m$, since then $T(\Gamma_k | \cdot; \boldsymbol{\alpha})$ is a two-dimensional tensor product of univariate Green's functions, and their well known properties yield

LEMMA 2.1. *In the case $k \leqslant m$*

(a) *$T(\Gamma_k | \mathbf{x}; \boldsymbol{\alpha})$ is infinitely differentiable at every $\mathbf{x} \in \mathbb{R}^2$ which is not of the form $\boldsymbol{\alpha} + t\mathbf{e}^j$, $t \in \mathbb{R}_+$, $j = 1, 2$.*

(b) *In points of the form $\mathbf{x} = \boldsymbol{\alpha} + t\mathbf{e}^j$ $(t > 0, 1 \leqslant j \leqslant 2)$, $T(\Gamma_k | \cdot; \boldsymbol{\alpha})$ has continuous derivatives up to order $l_j^k - 2$ where*

$$l_j^k := \#\{\mathbf{x}^i \mid 1 \leqslant i \leqslant k, \mathbf{x}^i \neq \mathbf{e}^j\}. \tag{2.5}$$

(c) *At $\mathbf{x} = \boldsymbol{\alpha}$, $T(\Gamma_k | \mathbf{x}; \boldsymbol{\alpha})$ has a zero of order $k - 2$, namely all its derivatives up to order $k - 3$ vanish at this point.*

The following theorem shows that the smoothness of $T(\Gamma_k | \cdot; \boldsymbol{\alpha})$ is determined by the directions $\{\mathbf{x}^j\}_{j=1}^n$, independent of the choice of the weight functions.

THEOREM 2.1. *For $m < k \leqslant n$ we have*

(a) *$T(\Gamma_k | \cdot; \boldsymbol{\alpha})$ is infinitely differentiable at all $\mathbf{x} \in \mathbb{R}^2$ which are not of the form $\boldsymbol{\alpha} + t\mathbf{e}^j$, $t \in \mathbb{R}_+$, $j = 1, 2, 3$.*

(b) *In points of the form $\mathbf{x} = \boldsymbol{\alpha} + t\mathbf{e}^j$ $(t > 0, \ ) \leqslant j \leqslant 3)$, $T(\Gamma_k | \cdot; \boldsymbol{\alpha})$ has continuous derivatives up to order $l_j^k - 2$ of all kinds, where l_j^k is defined as in* (2.5).

Proof. The proof of (a) and the proof of (b) for $j=1, 2$, in view of Lemma 2.1, are straightforward. For the case $j=3$ in (b) we start with $k=m+1$. Since by (2.2) the choice of $w_{m+1}(t)$ cannot affect the smoothness properties of $T(\Gamma_{m+1}|\cdot;\boldsymbol{\alpha})$ we choose $w_{m+1}(t)\equiv 1$. Now let $\mathbf{x}^0=\boldsymbol{\alpha}+t^0\mathbf{e}^3$ for some $t^0\in\mathbb{R}_+$. Clearly, by the choice $w_{m+1}(t)\equiv 1$, $D_{\mathbf{e}^3}T(\Gamma_{m+1}|\cdot;\ \boldsymbol{\alpha})=T(\Gamma_m|\cdot;\boldsymbol{\alpha})$, but by Lemma 2.1 $T(\Gamma_m|\cdot;\boldsymbol{\alpha})$ is infinitely differentiable at $\mathbf{x}^0$ or, when $t^0=0$, it is $m-3$ times continuously differentiable. Therefore in order to verify the smoothness it is enough to check derivatives only in the $(1, -1)$ direction. For this purpose we show that $\lim_{\mathbf{x}\to\mathbf{x}^0} D^j_{(1,-1)}\, T(\Gamma_{m+1}|\mathbf{x};\boldsymbol{\alpha})$ (for $1\leqslant j\leqslant m-2$) exists. First we choose $\mathbf{x}$ such that $x_1-x_2>x_1^0-x_2^0$. Note that in this case the ray with vertex $\mathbf{x}$ directed as $\mathbf{e}^3$ intersects the boundary of supp $T(\Gamma_m|\cdot;\boldsymbol{\alpha})$ at $(x_1-x_2+\alpha_2, \alpha_2)$. Hence according to (2.2)

$$T(\Gamma_{m+1}|\mathbf{x};\boldsymbol{\alpha})=\int_{(x_1-x_2/2+\alpha_2}^{(x_1+x_2)/2} T(\Gamma_m|\left(\frac{x_1-x_2}{2}+t, \frac{x_2-x_1}{2}+t\right);\boldsymbol{\alpha}\Big)\,dt. \tag{2.6}$$

By Lemma 2.1(c) $T(\Gamma_m|\cdot;\boldsymbol{\alpha})$ has vanishing derivative at $\boldsymbol{\alpha}$ up to order $m-3$ so in the case $x_1-x_2>x_1^0-x_2^0$ we get for $j=1, ..., m-2$

$$\begin{aligned}&D^j_{(1,-1)}T(\Gamma_{m+1}|\cdot;\boldsymbol{\alpha})\\&=\int_{(x_1-x_2)/2+\alpha_2}^{(x_1+x_2)/2} D^j_{(1,-1)}T\left(\Gamma_m\middle|\left(\frac{x_1-x_2}{2}+t, \frac{x_2-x_1}{2}+t\right);\alpha\right)dt+\text{other terms,}\end{aligned}$$

where the other terms vanish at $\mathbf{x}=\mathbf{x}^0$. Therefore

$$\begin{aligned}&\lim_{x\to\mathbf{x}^0} D^j_{(1,-1)}T(\Gamma_{m+1}|\cdot;\boldsymbol{\alpha})\\&\quad=\int_{(\alpha_1+\alpha_2)/2}^{(x_1^0+x_2^0)/2} D^j_{(1,-1)}T\left(\Gamma_m\middle|\left(\frac{x_1^0-x_2^0}{2}+t, \frac{x_2^0-x_1^0}{2}+t\right);\boldsymbol{\alpha}\right)dt.\end{aligned}$$

But the same expression is obtained when $\mathbf{x}$ is chosen such that $x_1-x_2<x_1^0-x_2^0$, so the smoothness at $\mathbf{x}^0$ for $k=m+1$ is verified. When $k>m+1$, we can assume, by induction, that $T(\Gamma_{k-1}|\cdot;\boldsymbol{\alpha})$ has continuous derivatives up to order $l_3^{k-1}-2=m-2$ at all points of the form $\boldsymbol{\alpha}+t\mathbf{e}^3$ $(t\in\mathbb{R})$ and therefore by (2.2) (since l_3^k-2 again equals $m-2$) we conclude that $T(\Gamma_k|\cdot;\boldsymbol{\alpha})$ has the desirable smoothness at $\mathbf{x}^0$. ∎

The explicit definition of $T(\Gamma_k|\cdot;\boldsymbol{\alpha})$ as given in (2.2) was useful for determining the smoothness of $T(\Gamma_k|\cdot;\boldsymbol{\alpha})$ but it is less convenient for later use. Therefore if we denote

$$\mathbf{v}^k=\begin{cases}\mathbf{x}^k, & \mathbf{x}^k=\mathbf{e}^j,\ 1\leqslant j\leqslant 2,\\ \frac{1}{2}\mathbf{x}^k, & \mathbf{x}^k=\mathbf{e}^3,\end{cases} \tag{2.7}$$

then (2.2) can be written in a unified way as

$$T(\Gamma_k\,|\,\mathbf{x};\boldsymbol{\alpha})=w_k(\mathbf{v}^k\cdot\mathbf{x})\int_{-\infty}^{\mathbf{v}^k\cdot\mathbf{x}} T(\Gamma_{k-1}\,|\,\mathbf{x}+(t-\mathbf{v}^k\cdot\mathbf{x})\,\mathbf{x}^k;\boldsymbol{\alpha})\,dt,$$

where "$\cdot$" denotes scalar product. A change of variables yields

$$T(\Gamma_k\,|\,\mathbf{x};\boldsymbol{\alpha})=w_k(\mathbf{v}^k\cdot x)\int_0^{\infty} T(\Gamma_{k-1}\,|\,\mathbf{x}-t\mathbf{x}^k;\boldsymbol{\alpha})\,dt. \tag{2.8}$$

This last formula corresponds to those of the "piecewise exponential" [DM_2, p. 271] and the truncated power [D; p. 183].

3. The Three Direction Periodic Exponential Box Spline

Let Γ_n be a defining set as in the previous section. Assume in addition that for $j=1,\ldots,n$

$$w_j(t)=e^{\rho_j t}r_j(t), \tag{3.1}$$

where $\rho_j\in\mathbb{R}$, $r_j(t+1)=r_j(t)$, $j=1,\ldots,m$, $r_j(t+\frac{1}{2})=r_j(t), j=m+1,\ldots,n$.

For such a defining set Γ_n(termed *periodic exponential defining set*) and a knot $\boldsymbol{\alpha}\in\mathbb{R}^2$, we construct here a three direction "periodic exponential box spline" (PEB-spline) and derive its basic properties. In particular it is shown that this box spline is a linear combination of certain truncated Tchebycheff functions, a fact which establishes its smoothness.

Definition 3.1. Let Γ_n be a periodic exponential defining set. The PEB-spline, based on Γ_k, $B(\Gamma_k\,|\,\mathbf{x};\boldsymbol{\alpha})$, is defined inductively as follows,

$$B(\Gamma_2\,|\,\mathbf{x};\boldsymbol{\alpha})=\begin{cases} w_1(x_1)\,w_2(x_2), & \mathbf{x}=\boldsymbol{\alpha}+\mathbf{t},\ \ \mathbf{t}\in[0,1)^2\\ 0, & \text{otherwise,}\end{cases} \tag{3.2}$$

and for $k\geqslant 3$

$$B(\Gamma_k\,|\,\mathbf{x};\boldsymbol{\alpha})=w_k(\mathbf{v}^k\cdot\mathbf{x})\int_0^1 B(\Gamma_{k-1}\,|\,\mathbf{x}-t\mathbf{x}^k;\boldsymbol{\alpha})\,dt, \tag{3.3}$$

where $\mathbf{v}^k$ is defined as in (2.7).

Example 3.1. When $w_j(t)\equiv 1, j=1,\ldots,n$, the PEB-spline is the three direction (polynomial) box spline (see [BD, BH]).

Example 3.2. When all weight functions are exponential then, in view of [R, Prop. 2.5], $B(\Gamma_k\,|\,\mathbf{x};\boldsymbol{\alpha})$, $k>m$, is a three direction exponential box spline. In fact it can be easily shown that all three direction exponential box splines can be produced by a suitable choice of exponential weight functions in Definition 3.1.

A simple change of variables in (3.3) yields

$$B(\Gamma_k \mid \mathbf{x}; \boldsymbol{\alpha}) = w_k(\mathbf{v}^k \cdot \mathbf{x}) \int_{-\mathbf{v}^k \cdot \mathbf{x}}^{-\mathbf{v}^k \cdot \mathbf{x}+1} B(\Gamma_{k-1} \mid \mathbf{x} - (\mathbf{v}^k \cdot \mathbf{x} + t)\, \mathbf{x}^k; \boldsymbol{\alpha})\, dt. \tag{3.4}$$

In the case $\mathbf{x}^k = \mathbf{e}^1$ (and therefore $\mathbf{v}^k = \mathbf{e}^1$), (3.4) becomes

$$B(\Gamma_k \mid \mathbf{x}; \boldsymbol{\alpha}) = w_k(x_1) \int_{x_1-1}^{x_1} B(\Gamma_{k-1} \mid (t_1; x_2); \boldsymbol{\alpha})\, dt,$$

which is a relation similar to (1.1).

The following result can be proved directly by using induction on k.

PROPOSITION 3.1. *$B(\Gamma_k \mid \cdot; \boldsymbol{\alpha})$, $2 \leqslant k \leqslant n$, is a nonnegative function which is positive in the interior of its support. This support is independent of $\{w_j(t)\}_{j=1}^k$, hence it coincides with that of the corresponding polynomial box spline.*

The support is the image of $[0, 1]^n$ under the transformation $\mathbf{t} \to \sum_{j=1}^n t_j \mathbf{x}^j$. For details see [BH, p. 15].

PROPOSITION 3.2. *The PEB-spline is translation invariant in the following sense,*

$$B(\Gamma_k \mid \mathbf{x} + \boldsymbol{\delta}; \boldsymbol{\alpha} + \boldsymbol{\delta}) = C_{\Gamma_k}(\boldsymbol{\delta})\, B(\Gamma_k \mid \mathbf{x}; \boldsymbol{\alpha}), \qquad \boldsymbol{\delta} \in \mathbb{Z}^2, 2 \leqslant k \leqslant n, \tag{3.5}$$

with

$$C_{\Gamma_k}(\boldsymbol{\delta}) = \prod_{j=1}^{k} e^{\rho_j \boldsymbol{\delta} \cdot \mathbf{v}^j}. \tag{3.6}$$

Proof. For $k = 2$, (3.5) can be directly verified from (3.1) and (3.2). Assume that $k > 2$ and that the claim is valid for $k - 1$. Let $\boldsymbol{\alpha} \in \mathbb{Z}^2$; then by (3.3) and the induction hypothesis

$$\begin{aligned} B(\Gamma_k \mid \mathbf{x} + \boldsymbol{\delta}; \boldsymbol{\alpha} + \boldsymbol{\delta}) &= w_k(\mathbf{v}^k \cdot (\mathbf{x} + \boldsymbol{\delta})) \int_0^1 B(\Gamma_{k-1} \mid \mathbf{x} + \boldsymbol{\delta} - t\mathbf{x}^k; \boldsymbol{\alpha} + \boldsymbol{\delta})\, dt \\ &= e^{\rho_k \mathbf{v}^k \cdot \boldsymbol{\delta}} w_k(\mathbf{v}^k \cdot \mathbf{x})\, C_{\Gamma_{k-1}}(\boldsymbol{\delta}) \int_0^1 B(\Gamma_{k-1} \mid \mathbf{x} - t\mathbf{x}^k; \boldsymbol{\alpha})\, dt \\ &= C_{\Gamma_k}(\boldsymbol{\delta})\, B(\Gamma_k \mid \mathbf{x}; \boldsymbol{\alpha}). \quad \blacksquare \end{aligned}$$

The following difference operators are useful for establishing the connection between the PEB-spline and the corresponding truncated Tchebycheff,

$$\Delta^j f(\cdot) := f(\cdot) - C_{\Gamma_{j-1}}(-\mathbf{x}^j) f(\cdot + \mathbf{x}^j), \qquad j = 1, ..., n, \tag{3.7}$$

$$\Delta^{\Gamma_k} f(\cdot) = \prod_{j=1}^{k} \Delta^j f(\cdot), \tag{3.8}$$

where $C_{\Gamma_j}, j = 1, ..., n-1$, are defined as in (3.6) and $C_{\Gamma_0} := 1$.

THEOREM 3.1.

$$B(\Gamma_k \mid \mathbf{x}; \boldsymbol{\alpha}) = \Delta^{\Gamma_k} T(\Gamma_k \mid \mathbf{x}; \cdot)|_{\boldsymbol{\alpha}}, \qquad 2 \leqslant k \leqslant n. \tag{3.9}$$

Proof. By induction on k. First note that $C_{\Gamma_0}(-\mathbf{x}^1) = C_{\Gamma_1}(-\mathbf{x}^2) = 1$ so the case $k = 2$ is easily obtained by comparing (2.1) and (3.2). Assume $k > 2$, then by (2.8) and the induction hypothesis

$$\begin{aligned} \Delta^{\Gamma_k} T(\Gamma_k \mid \mathbf{x}; \cdot)|_{\boldsymbol{\alpha}} &= \Delta^k w_k(\mathbf{v}^k \cdot \mathbf{x}) \int_0^\infty B(\Gamma_{k-1} \mid \mathbf{x} - t\mathbf{x}^k; \cdot)|_{\boldsymbol{\alpha}}\, dt \\ &= w_k(\mathbf{v}^k \cdot \mathbf{x}) \left[\int_0^\infty B[\Gamma_{k-1} \mid \mathbf{x} - t\mathbf{x}^k; \boldsymbol{\alpha})\, dt \right. \\ &\quad \left. - C_{\Gamma_{k-1}}(-\mathbf{x}^k) \int_0^\infty B(\Gamma_{k-1} \mid \mathbf{x} - t\mathbf{x}^k; \boldsymbol{\alpha} + \mathbf{x}^k)\, dt \right]. \end{aligned}$$

But by Proposition 3.2

$$\begin{aligned} \int_0^\infty B(\Gamma_{k-1} \mid \mathbf{x} - t\mathbf{x}^k; \boldsymbol{\alpha} + x^k)\, dt &= C_{\Gamma_{k-1}}(\mathbf{x}^k) \int_0^\infty B(\Gamma_{k-1} \mid \mathbf{x} - (t+1)\, \mathbf{x}^k; \boldsymbol{\alpha})\, dt \\ &= C_{\Gamma_{k-1}}(\mathbf{x}^k) \int_1^\infty B(\Gamma_{k-1} \mid \mathbf{x} - t\mathbf{x}^k; \boldsymbol{\alpha})\, dt \end{aligned}$$

which in view of (3.3) establishes (3.9). ∎

Combining Theorem 3.1 with Theorem 2.1 one has

COROLLARY 3.1. *For* $\mathbf{x} = \boldsymbol{\alpha} + \boldsymbol{\delta} + t\mathbf{e}^j$, $\boldsymbol{\delta} \in \mathbb{Z}^2$, $1 \leqslant j \leqslant 3$, $t \in \mathbb{R}$, $B(\Gamma_k \mid \cdot\,; \boldsymbol{\alpha})$ *has continuous derivatives up to order* $l_j^k - 2$ at $\mathbf{x}$, *where* l_j^k *is as in* (2.5). *For all other* $\mathbf{x} \in \mathbb{R}^2$, $B(\Gamma_k \mid \cdot\,; \boldsymbol{\alpha})$ *is infinitely differentiable at* $\mathbf{x}$.

In other words the smoothness of the three direction PEB-spline coincides with that of the polynomial box spline based on the same direction set (see [BH, p. 14]).

4. A Recurrence Relation for PEB-Splines

Inspired by observations previously made in [DM$_2$] and [R], we derived in [DR$_2$] recurrence relations for certain univariate Tchebycheffian B-splines. In particular it was shown there that, if the last weight function used in the generating process of the Tchebycheffian B-spline is exponential, then this B-spline is a linear combination with constant coefficients of four lower order B-splines. Here, we give a modification of this recurrence relation for the three direction PEB-spline, thus demonstrating that the univariate setting for the result mentioned above is of no particular significance.

The following proposition is analogous to Proposition 2.2 and is obtained directly from (3.4):

Proposition 4.1. *Let $m<k\leqslant n$ and let D^k be as in* (2.3); *then*

$$D^k B(\Gamma_k\,|\,\cdot\,;\boldsymbol{\alpha})|_{\mathbf{x}} = w_k(\mathbf{v}^k\cdot\mathbf{x})[B(\Gamma_{k-1}\,|\,\mathbf{x};\boldsymbol{\alpha}) - B(\Gamma_{k-1}\,|\,\mathbf{x}-x^k;\boldsymbol{\alpha})]. \tag{4.1}$$

For a special choice of $w_k(t)$ and $w_{k-1}(t)$ another result is valid:

Lemma 4.1. *Assume $w_k(t)\equiv 1$, $w_{k-1}(t)=e^{\rho t}$, $k>m+1$; then*

$$(D_{\mathbf{x}^k}-\rho)\,B(\Gamma_k\,|\,\cdot\,;\boldsymbol{\alpha})|_{\mathbf{x}} = B(\tilde{\Gamma}_{k-1}\,|\,\mathbf{x};\boldsymbol{\alpha}) - e^{\rho}B(\tilde{\Gamma}_{k-1}\,|\,\mathbf{x}-\mathbf{x}^k;\boldsymbol{\alpha}), \tag{4.2}$$

where $\tilde{\Gamma}_{k-1}$ is obtained from Γ_{k-1} by replacing $w_{k-1}(t), w_{k-2}(t)$ by 1, $e^{\rho t}w_{k-2}(t)$, respectively.

Proof. Since $k>m+1$, $\mathbf{x}^k=\mathbf{x}^{k-1}$, $\mathbf{v}^k=\mathbf{v}^{k-1}$ and so by combining (3.3) and (3.4) we get

$$B(\Gamma_k\,|\,\mathbf{x};\boldsymbol{\alpha}) = e^{\rho\mathbf{v}^k\cdot\mathbf{x}}\int_0^1 e^{-\rho t}\int_{-\mathbf{v}^k\cdot\mathbf{x}+t}^{-\mathbf{v}^k\cdot\mathbf{x}+t+1} B(\Gamma_{k-2}\,|\,\mathbf{x}-(\mathbf{v}^k\cdot\mathbf{x}+\tau)\,\mathbf{x}^k;\boldsymbol{\alpha})\,d\tau\,dt.$$

Therefore since $D_{\mathbf{x}^k}[\mathbf{x}-(\mathbf{v}^k\cdot\mathbf{x}+\tau)\,\mathbf{x}^k]=0$

$$\begin{aligned}(D_{\mathbf{x}^k}-\rho)\,B(\Gamma_k\,|\,\mathbf{x};\boldsymbol{\alpha}) &= e^{\rho\mathbf{v}^k\cdot\mathbf{x}}\left[\int_0^1 e^{-\rho t}B(\Gamma_{k-2}\,|\,\mathbf{x}-t\mathbf{x}^k;\boldsymbol{\alpha})\,dt\right.\\ &\qquad\left.-\int_0^1 e^{-\rho t}\,B(\Gamma_{k-2}\,|\,\mathbf{x}-\mathbf{x}^k-t\mathbf{x}^k;\boldsymbol{\alpha})\,dt\right]\\ &= B(\tilde{\Gamma}_{k-1}\,|\,\mathbf{x};\boldsymbol{\alpha}) - e^{\rho}B(\tilde{\Gamma}_{k-1}\,|\,\mathbf{x}-\mathbf{x}^k;\boldsymbol{\alpha}). \quad \blacksquare\end{aligned}$$

Subtracting (4.2) from (4.1) with $w_k\equiv 1$ and then multiplying both sides by $w_k(\mathbf{v}^k\cdot\mathbf{x})$, we obtain

THEOREM 4.1. *Assume* $w_{k-1}(t)=e^{\rho t}$, $\rho \neq 0, k > m+1$; *then*

$$B(\Gamma_k \mid \mathbf{x}; \boldsymbol{\alpha}) = \rho^{-1} w_k(\mathbf{v}^k \cdot \mathbf{x})[B(\Gamma_{k-1} \mid \mathbf{x}; \boldsymbol{\alpha}) - B(\Gamma_{k-1} \mid \mathbf{x}-\mathbf{x}^k; \boldsymbol{\alpha}) - B(\tilde{\Gamma}_{k-1} \mid \mathbf{x}; \boldsymbol{\alpha}) + e^{\rho} B(\tilde{\Gamma}_{k-1} \mid \mathbf{x}-\mathbf{x}^k; \boldsymbol{\alpha})],$$

where $\tilde{\Gamma}_{k-1}$ *is defined as in Lemma* 4.1.

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